

# Defining quantum computational complexity over continuous variables

Saeed Mehraban

Joint work with U. Chabaud (INRIA), Sevag Gharibian (Paderborne), Arsalan Motamedi (U Waterloo), H. Naeij (Paderborne), D. Rudolph (Paderborne) and D. Sambrani (Paderborne).

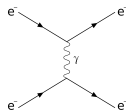
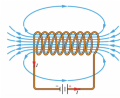
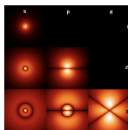
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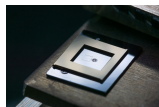
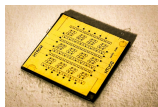


# Motivation

- Many physical systems are continuous variables.
  - Position and momenta (e.g., Oscillators, Molecular systems LC circuits)
  - Bose-Einstein Condensates
  - E.M. amplitudes
  - Quantum Field Theories



- Bosonic computational devices (Photonic and superconducting circuits, Optomechanical devices)



- Computation over real numbers

# What is this talk about?

- **Main problem:** Defining a model of quantum computation over continuous variables (CV).
- The state space of CV systems such as Bosonic systems is infinite dimensional
- **Today:** We explain several **challenges** in defining a **bosonic computational model** due to the infinite dimensions.
- Throughout this talk  
Bosonic systems = Infinite-dim systems = Continuous-variable systems

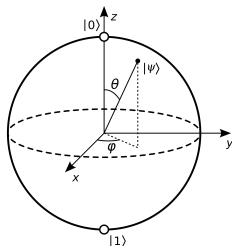
# Discrete v.s. continuous variables

DV	CV
Qbits/ Qdits (Spins, polarization)	Qmodes (Position, Momentum, Particle number)
Unit vectors in $\mathbb{C}^d$	Unit vectors in $\mathbb{C}^\infty$ (e.q. Square integrable functions)
Observables have discrete and bounded spectra	Observables may have bounded, unbounded, discrete or continuous spectra
Clifford/non-Clifford	Gaussian/non-Gaussian
Universality and Fast Compiling	??? (this talk)
Quantum complexity class (BQP)	??? (this talk)

# Quantum states

- **Qubit:** The state of a two level-system quantum is specified by  $|\psi\rangle \in \mathbb{C}^2$
- $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ ,  $\alpha, \beta \in \mathbb{C}$  are called amplitudes  $|\alpha|^2 + |\beta|^2 = 1$ . Upon measurement

$$Pr(0) = |\alpha|^2, \quad Pr(1) = |\beta|^2.$$



- **General quantum states:**  $|\psi\rangle \in \mathbb{C}^d$  for  $1 \leq d \leq \infty$ ,  $|\psi\rangle = \sum_j \alpha_j |j\rangle$  with  $\sum_j |\alpha_j|^2 = 1$ . Upon measurement we obtain

$$Pr(j) = |\alpha_j|^2.$$

# Quantum operations

- $U \in \mathbb{C}^{d \times d}$  is a **unitary matrix** if  $U^\dagger = U^{-1}$  where  $U_{jk}^\dagger = U_{kj}^*$ . It is a **generalization of orthogonal matrices** to the complex number field.
- The evolution of a closed system is according to a **unitary matrix** (which preserves the law of probability).

$$U : |\psi\rangle \in \mathbb{C}^d \mapsto U|\psi\rangle$$

- Basic unitary **gates**:

$$\text{Pauli: } X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\text{Clifford: } H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\text{Non-Clifford: } T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

# Universality and Fast Compilation

- A gate set is **universal** if any unitary operation can be approximated to arbitrary precision using a finite sequence of gates from that gate set.
- Starting with a simple state Clifford (which includes Pauli) is classically simulable
- However Clifford +  $T$  is universal
- **Solovay Kitaev:**

## Theorem (Solovay–Kitaev Theorem)

*Let  $\mathcal{G}$  be a finite set of single- and two-qubit gates that generates a dense subgroup of  $SU(2^n)$ . Then there exists a universal constant  $c$  such that any unitary  $U \in SU(2^n)$  can be approximated to accuracy  $\varepsilon > 0$  by a sequence of gates from  $\mathcal{G}$  of length  $O(\log^c \frac{1}{\varepsilon})$ .*

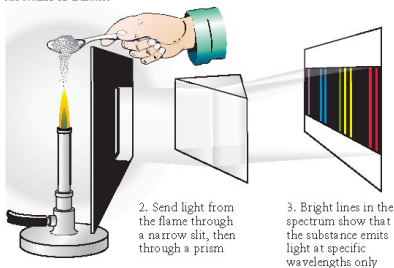
# Physical observables

- Physical observables correspond to Hermitian operators (which have real spectrum).  
 $O = O^\dagger$ ,

$$O |v_j\rangle = o_j |v_j\rangle, o_j \in \mathbb{R}.$$

- If the state of a system is  $|v_j\rangle$ , the observable will **deterministically** reveal  $o_j$ .
- For general  $|\psi\rangle$ , a measurement of  $O$  will reveal expectation value  $\langle\psi| O |\psi\rangle$ .
- In finite dimensions observables are **bounded with discrete (quantized) spectrum**.

1. Add a chemical substance to a flame



- Example:** Physical spins:  $S_x = \frac{\hbar}{2} X$ ,  $S_y = \frac{\hbar}{2} Y$ ,  $S_z = \frac{\hbar}{2} Z$ .

# Quantum states

- We can view quantum states in different basis

- **Position basis:**  $\psi \in \mathcal{L}^2(\mathbb{R})$

$$\psi(x), \quad \int_{\mathbb{R}} |\psi(x)|^2 dx = 1,$$

or momentum basis via Fourier transform!

- **Particle number (Fock) basis:**<sup>1</sup>  $|\psi\rangle \in \ell^2(\mathbb{C})$

$$|\psi\rangle = \sum_{n \geq 0} \alpha_n |n\rangle, \quad \sum_{n \geq 0} |\alpha_n|^2 = 1$$

where  $\{|n\rangle, n \geq 0\}$  is known as the Fock or particle number basis.

- Coherent states

$$|z\rangle = e^{-|z|^2/2} \sum_{n \geq 0} \frac{z^n}{\sqrt{n!}} |n\rangle, \quad z \in \mathbb{C}$$

<sup>1</sup> $\psi(x) = \sum_{n \geq 0} \alpha_n H_n(x)$ ,  $H_n$  Hermite polynomials.

# Unbounded operators

- **Position** ( $X$ ) and **momentum** ( $P$ ) have continuous eigenbasis

$$X|x\rangle = x|x\rangle, x \in \mathbb{R}, \quad P|p\rangle = p|p\rangle, p \in \mathbb{R}$$

with algebra

$$[X, P] = i\hbar$$

They are Fourier dual!

- **Particle number:**  $N = \frac{1}{2}(X^2 + P^2 - I)$  has discrete spectrum

$$N|n\rangle = n|n\rangle, n \geq 0.$$

- **Multiple modes:**  $O_j$  is operator  $O$  on mode  $j$ .

# Gaussian gates

- Consider **polynomial hamiltonians**  $H$  of deg  $k$  in  $(X_1, \dots, P_1, \dots)$ ,  $H = H^\dagger$ .
- $U = e^{iH}$  is a Gaussian unitary if hamiltonian  $H$  has deg  $\leq 2$  in  $X_i$  and  $P_j$ 
  - Displacement  $e^{i(aX+bP)}$
  - Rotation  $e^{i\theta N}$
  - Squeezing  $e^{ir(XP+PX)}$
  - Passive linear optical elements: Beam splitters and phase shifters.
- A quantum state  $|\psi\rangle$  is called Gaussian if  $|\psi\rangle = U|0\rangle$  for Gaussian  $U$ .
- A specific Gaussian gate is  $F = e^{i\frac{\pi}{4}(X^2+P^2)}$  which is called Fourier transform because

$$FXF^{-1} = P, \quad FPF^{-1} = -X,$$

# Gaussian gates

## Gaussian operations are easy to represent:

If  $U$  is Gaussian

- **Single mode:**

$$U = \text{Rotation} \times \text{Displacement} \times \text{Squeezing}$$

- **Multiple modes:**

$$U = V \bigotimes_i G_i W$$

$V, W$  are passive linear optical and  $G_i$  are single mode Gaussian

## Theorem

*Starting with Gaussian states, Gaussian circuits can be efficiently simulated in polynomial time.*

This can be viewed as a CV generalization of the Gottesman-Knill theorem for (DV) Clifford circuits.

# Non-Gaussian gates

- **The cubic gate:**  $e^{iX^3}$ . Was used by Gottesman-Kitaev-Preskill ('00) to robustly encode universal DV quantum computations inside DV degrees of freedom.
- Along with Gaussian operations the cubic gate may lead to doubly exponentially fast energy growths (CJMM 24 + this talk)
- **The Kerr gate:**  $e^{i(X^2+P^2)^2}$ . From nonlinear material with **intensity dependent refractive index**. Preserves boson number.
- **Oscillator controlled Gaussian gates:** E.g. controlled squeezing  $e^{-iX_1(a_2^2+a_2^{\dagger 2})t}$
- Other examples include CV-DV interactions (e.g. qubit controlled Gaussian operations, Jaynes Cummings interaction)

# Stellar rank

- For a CV quantum state  $|\psi\rangle \in \mathbb{C}^\infty$ , we can assign a Holomorphic “stellar” function

$$F_\psi(z) = e^{\frac{|z|^2}{2}} \langle z^* | \psi \rangle$$

Normalization:  $\int_{\mathbb{C}} |F_\psi(z)|^2 d\mu(z) = 1$ , where  $d\mu(z) = \frac{e^{-|z|^2}}{\pi} d^2z$  is the Gaussian measure.

- Any stellar function can be decomposed as

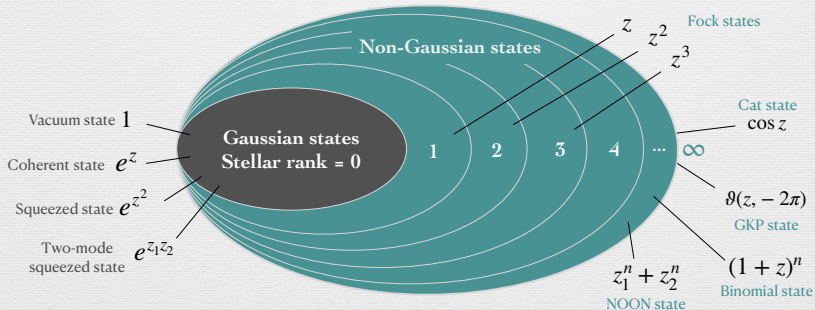
$$F_\psi(z) = P_r(z) \times G(z).$$

where  $G(z)$  is a Gaussian function and  $P$  is a polynomial of degree  $0 \leq r \leq \infty$

- $r$  is called the stellar rank of  $|\psi\rangle$ . It measures “non-Gaussianity”.

## Multiple systems

## Multimode non-Gaussian hierarchy



# Defining a CV Quantum Computational Model

- **Lloyd and Braunstein '98** introduced a model of quantum computation based on bosonic degrees of freedom.
- They considered the gateset **Kerr + Gaussian**
- They suggested the following universality result
  - “**The Gaussian + Kerr gate set is universal:** Any Hamiltonian as a **polynomial in position and momenta operations** may be approximated to within arbitrary precision using sequences of Kerr and Gaussian gates.
- Soon it was observed that due to infinite dimensions standard techniques does not work and the above statement is not obvious (E.g. usual Trotter bounds rely on bounded operators)
- Several ideas have been introduced e.g.
  - State dependent Trotter bounds
  - Exact decomposition of gates into each other
  - More recently polynomial Hamiltonians were shown to be universal for unitary matrices that allow “effective” DV approximation

# Unusual complexity of CV computations

Recently Brenner, Caha, Coiteux-Roy and Koenig (2024) showed

## Theorem

*There exists a quantum algorithm that can **factor (exponentially large) integers** in polynomial time using only **three oscillators** and one qubit.*

How is that possible?

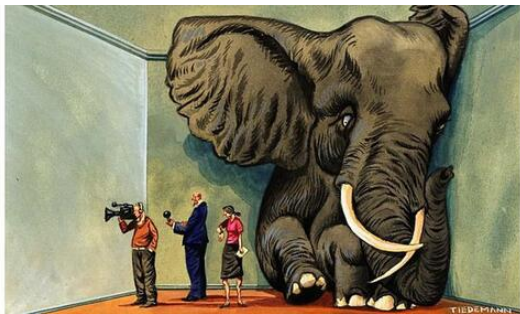
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Answer: Energy (Elephant in the room)

# Why Energy Matters

- The outlined factoring algorithm consumes **an exponential amount of energy**
- Energy (average photon number used in the computation) plays the role of a computational resource.
- **Question:** How does increasing energy affect computational power of a Bosonic computational model?
- We answer this question in **this talk**:
  - 1 **Energy growth:** CV computations using polynomial Hamiltonians can **generate up to infinite (!)** amount of energy in finite time
  - 2 **Complexity lower bounds:** This energy production can be used to simulate towers of exponential time computations.
  - 3 **Complexity upper bounds:** Some gate sets can be simulated in smaller complexity classes. This implies certain **no go theorems for fast compilation** of polynomial gates into each others.

# Energy Growth Phenomena

Not all gate sets are equivalent in terms of energy production

## Theorem (Informal)

- 1 Gaussian + Kerr gate: energy  $\leq e^{O(t)}$ .
- 2 Gaussian + cubic phase: energy  $\sim (2^t)^{2^t}$   
(This is true even with a constant rate of dissipation.)
- 3 Certain two-mode Hamiltonians (such as  $iX_1^4(a_2^2 - a_2^{\dagger 2})$ ) yield  $\infty$  energy in constant time (starting with the vacuum state).

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**Remark on (3):** There are natural normalized quantum states with infinite expected energy. However, for any normalized quantum state infinite energy there exists a quantum state that is close within statistical distance that has finite energy. E.g.

$$|\phi\rangle = \frac{\sqrt{6}}{\pi} \sum_{n \geq 1} \frac{1}{n} |n\rangle$$

We see  $\langle \phi | N | \phi \rangle = \frac{6}{\pi^2} \sum_{n \geq 1} 1/n = \infty$ . But

$$|\phi\rangle \approx |\phi'\rangle \propto |1\rangle + 1/2 |2\rangle + 1/3 |3\rangle + 1/4 |4\rangle + 1/5 |5\rangle + 1/6 |6\rangle$$

# Intuition for cubic gate energy growth

Consider the following process

$$\underbrace{e^{i\hat{X}^3} F e^{i\hat{X}^3} F \dots e^{i\hat{X}^3} F}_{t \text{ times}}$$

We observe this process leads to a repeated squaring phenomenon:

- $P \xrightarrow{X^3/3} P + X^2 \xrightarrow{F} -X + P^2 \xrightarrow{X^3/3} -X + (P + X^2)^2$
- If we continue this  $t$  rounds we get **repeated squaring**, i.e., the leading term would be  $X^{2^t}$ .
- The expected number of particles in the system may grow **doubly exponentially fast!**

## Theorem (This work)

*Expected particle number in cubic + Gaussian circuits with cubic depth  $t$  may grow at least (and up to) doubly exponentially fast in  $t$ .*

Previous work CJMM 24 implied an EXPSPACE simulation for this model.

# Energy Implies Computational Complexity

Define the notation:

$$\exp^{(k)}(n) = \underbrace{\exp(\exp(\dots \exp(n)\dots))}_{k \text{ times}}$$

And define

$$a \uparrow\uparrow b = \underbrace{a^{a^{\dots^a}}}_{b \text{ times}}$$

and

$$\text{PTOWER} := \cup_{k \in \mathbb{N}} \text{DTIME}(2 \uparrow\uparrow n^k)$$

**Theorem (Informal)** For suitable finite gate sets  $\mathcal{G}$ :

- 1 **NP**  $\subseteq$  **CVBQP** $[\mathcal{G}]$  with exponential energy,  $O(1)$  modes.
- 2 **NTIME** $(\exp^{(k)}(n)) \subseteq$  **CVBQP** $[\mathcal{G}]$  with  $O(k)$  modes.
- 3 **PTOWER**  $\subseteq$  **CVBQP** $[\mathcal{G}]$ .

Item (3) shows that using constant number of oscillators we can even get NP (compare with the factoring result).

# Energy Hierarchy Theorem

Does more energy always help?

**A more fine grained result:** For any  $\epsilon > 0$  there exists an (oracle) problem that can be solved using energy  $1/\epsilon^2$  and not energy  $o(1/\epsilon)$ .

## Implication

Even at smaller amounts of energy, more energy **provably yields** more computational power.

# Upper Bounds and Simulation

- 1 If we are promised that a CV computation consumes at most polynomial energy then we can simulate this model in  $\mathbf{BQP}_{\text{poly}}$  (polynomial advice).
- 2 “Physical” CV computations (finite energy) are decidable.
- 3  $\mathbf{CVBQP}[X^3]$  with exponential energy  $\subseteq \mathbf{PP}$ .

The previous bound for (3) was  $\mathbf{PSPACE}$ . Now:  $\mathbf{PP}$  — closer to  $\mathbf{BQP}$ .

**Question:** Can we know beforehand if a CV computation will use a polynomial amount of energy?

# Undecidability Results

We show:

Computing basic properties of CV systems can itself be undecidable.

**Theorem:** It is undecidable whether for a constant degree  $H$

- ① A Gaussian state evolved under  $H$  has finite energy.
- ② The evolution of an essentially self-adjoint  $H$  preserves the Schwartz space.
- ③ A symmetric polynomial  $H$  is essentially self-adjoint.

\* **Essentially self-adjoint:** Hermitian conjugate is well defined in the same domain as the operator itself. Essential in order to have a well-defined quantum dynamics.

\* **Schwartz space:** (informal) states which have finite polynomial observable expectations.

This raises an important challenge in formulating CV computations:

While we hope to define stable computations without energy divergence, we cannot decide whether a given gate set leads to stable computation or not.

# No-go for CV Solovay–Kitaev

- **Recall DV Solovay–Kitaev:** Efficient universal approximation with polylog overhead.
- **CV analogue fails:** For some gate set we can generate towers of complexity classes (i.e. PTOWER) within polynomial time, while for some other gate set such as cubic with Gaussian we get  $\subseteq \text{EXPSPACE} \subsetneq \text{PTOWER}$ .
- **Therefore:** a fast compilation of one gate set into another will collapse these complexity classes (which we know it is not true)

**Remark:** Therefore we need to come up with **energy constrained versions** of SK. However as we discussed, reasoning about energy consumption itself is an undecidable problem.

# Steps of the hardness reduction

We do this in three main steps

- 1 Use controlled squeezers to create an extremely high energy state
- 2 We use **adiabatic computation** to create a state which encodes the solution “diophantine equation” (which is known to be hard, even undecidable)
- 3 We decode the solution

\* Item (2) involves significant reasoning about domain of unbounded operators.

# Hardness of Diophantine equations

We perform reduction from solving Diophantine equations (aka Hilbert's 10th problem):

**Theorem (Matiyasevich–Robinson–Davis–Putnam '70)**

*It is undecidable whether a multivariate polynomial  $F \in \mathbb{Z}[x_1, \dots, x_n]$  has an integer solution.*

**Theorem (Manders and Adleman '78)**

*The language  $\{(\alpha, \beta, \gamma) \in \mathbb{N}_0^3 \mid \exists x_1, x_2 \in \mathbb{N}_0: \alpha x_1^2 + \beta x_2 - \gamma = 0\}$  is NP-complete.*

**Theorem (Manders and Adleman '78)**

*There exists an integer polynomial  $F(x_1, \dots, x_k)$ , such that the promise problem  $A^{(F,E)}$  is  $\text{NTIME}(T(n))$ -hard for  $E(n) = 2^{2^{10T^2(n)}}$ :*

$$A_{\text{yes}}^{(F,E)} = \{x \mid \exists x_2, \dots, x_k \in \{0, \dots, E(\lceil \log x \rceil)\}: F(x, x_2, \dots, x_k) = 0\}, \quad (1a)$$

$$A_{\text{no}}^{(F,E)} = \{x \mid \forall x_2, \dots, x_k \in \mathbb{N}_0: F(x, x_2, \dots, x_k) \neq 0\}, \quad (1b)$$

# Decoding: Measuring a bit from a high energy state

- Define the following Hamiltonian with free parameter  $c$ :

$$G^c = ca^\dagger a + i/2(a^{\dagger 2} - a^2)$$

- $G^c$  is the Hamiltonian of the detuned degenerate parametric amplifier
- Main feature:

## Theorem

*For  $c > 1$  the dynamics are oscillatory (bounded energy) and for  $c < 1$  exponential ( $e^{-itG^{(0)}} |0\rangle = S(-t) |0\rangle$ ).*

- We can distinguish the two cases, hence decide whether  $c < 1$  or  $c > 1$ .

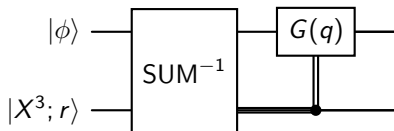
# BQP upper bound for computations with polynomial energy

- 1 Use a technique called “gentle measurement lemma” to truncate CV computations. However truncated operations are not unitary (they are close to unitaries tho)
- 2 Use a technique known as “block encoding” to simulate the truncated CV operations using unitary gates.

# PP upper bound on cubic + Gaussian

- 1 We use magic gate injections to reformulate a CV computation with  $X^3$  + Gaussian gates as cubic (resource) states followed by Gaussian computations. A cubic state is

$$|X^3; r\rangle = e^{iX^3} S(r) |0\rangle$$



- 2 Decompose cubic states into exponentially many Gaussian states

$$|X^3; r\rangle^{\otimes m} = \sum_{j=1}^R c_j |G_j\rangle$$

- 3 Compute each Gaussian branch using a CV Gottesman-Knill theorem.

# Discussion and Open Questions

- What happens in DV–CV hybrid models?
- Can we simulate Gaussian +  $X^3$  gate set within PP without energy bounds?  
Can this model simulate BQP?
- Can we formulate a model of computation where energy is naturally under control and we get universality and compiling?
- What does this result imply for computability aspects of quantum field theories?
- What do these results imply for fault tolerance in CV computations?
- Foundations for simulation of CV systems on CV devices.

# Summary

- Need for a Bosonic model of computation with energy is a **computational resource**.
- We show high energy  $\Rightarrow$  enormous computational power. A priori it is not clear if energy higher than a certain amount would help with encoding complex problems.
- Reasoning about energy growth in a CV system is itself undecidable.
- Bosonic models of computation may depend on the choice of gate sets (Unlike the DV setting). Certain upper bound results for some gate sets but not the others.

**Thank you!**